# Designing for Structural Reliability

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**A design philosophy based upon the probable variations in the applied loads, temperatures, geometry, materials and strength is presented. Fundamental relationships are reviewed indicating how the statistical nature of the design problem can be translated into logical safety factors. Arbitrary safety factors can result in unnecessary weight or inadequate reliability. Examples are presented, demonstrating how to employ available data to extrapolate for untested conditions and to design for a desired structural reliability.**

## Nomenclature

$a$ = ratio of coefficients of variability ($\gamma_a/\gamma_f$)
$f$ = frequency distribution of variate
$m$ = stability exponent
$n$ = number of components or tests
$r$ = factor of safety ($\bar{x}_f/\bar{x}_a$)
$t$ = thickness
$u$ = nondimensional deviation from the mean $[(y - \bar{y})/\sigma_y = \phi^{-1}(u)]$
$x$ = variate
$x_a$ = applied index
$x_f$ = failure index
$y$ = difference of failure and applied indices ($x_f - x_a$)
$A$ = area
$C$ = constant
$C$ = confidence
$D$ = damage index
$E$ = modulus of elasticity
$N$ = axial load per inch
$Pr$ = probability
$R$ = test ratio of failure to design indices
$R$ = radius
$S$ = stress
$T$ = temperature
$\gamma$ = coefficient of variability ($\sigma_x/\bar{x}$)
$\delta x_i$ = variation from the mean ($x_i - \bar{x}$)
$\xi$ = deviation parameter ($u\ \gamma_f$)
$\sigma$ = standard deviation
$\phi$ = probability of failure
$\phi(u_0)$ = cumulative standardized normal distribution $\left[\left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{u_0} e^{-u^2/2}\,du\right]$
$(\bar{\ \ })$ = bars over variate represents mean values

### *Subscripts*

$a$ = applied
$f$ = failure
$A$ = $A$ values in Ref. 5; $\phi(u) = 0.01$, $u = -2.327$
$B$ = $B$ values in Ref. 5; $\phi(u) = 0.1$, $u = -1.287$
$C$ = confidence level
$R$ = room temperature
$T$ = temperature

THE prime purpose of this presentation is to introduce the designer to a recent philosophy, which is being considered more and more as the design problems become exceedingly complex. The presentation does not intend to suggest that past design procedures be thrown out in favor of a probabilistic approach. It does suggest, however, that the designer should be cognizant of both philosophies. This will enable him to compare the merits of each in any application, depending upon the amount of available data, and to estimate the probabilistic characteristics of the design.

A prime requirement of an efficient design is to select proper factors of safety. Too large a factor of safety would result in increased weight and poorer performance, whereas too small a factor would result in inadequate structural reliability.

The structural design is a probabilistic problem that should consider the desired structural reliability. Reference 1 indicates that the over-all structural reliability should be distributed among the components of a structure so that the probability of failure of each component is proportional to its weight. After a favorable distribution of the probability of failure has been established, it is then necessary to design each component with the proper safety factor to attain this goal. The safety factor, defined herein as the ratio of the probable strength to the probable critical force, is a function of the desired reliability and the variations in the applied loads and strengths. The factor of safety, more aptly called a factor of ignorance, is no guarantee of safety but only a statistical quantity, which will, if properly chosen, result in an efficient structure. The factor of safety should not be invariant or arbitrary but should be selected with careful deliberation. If the applied and failing loads are sharply defined (insignificant variability) then a safety factor of 1.01 would result in extremely high reliability. On the other hand, if the variability is high then a large safety factor would be required to result in an insignificant probability of failure.

Present safety factors are based upon past experiences and engineering judgment. Unfortunately the structural requirements have become more stringent in modern vehicles requiring the designer to consider new failure criteria (e.g., fatigue, creep), new materials and constructions (e.g., brittle materials, fibrous weaves), and more complex loading conditions (e.g., temperature-load histories). This has resulted in greater variability in the applied and failing loads than had been encountered in the past. The effect has been to increase the probability of failure for structures designed for the arbitrary standardized factors of safety. Past experiences and engineering judgment, although of high value, is not sufficient to properly assess the proper safety factors and can result in structures whose reliability is significantly different from the optimum reliability.

The factor of safety should be "custom fitted" to each design. It should increase with increases in the desired reliability and in the variability of the applied and failing loads. Employing a constant arbitrary factor of safety would usually result in haphazard structural reliabilities for each component. This is not efficient, because weight would be wasted in those components that had a greater share of the structural reliability then they were entitled to by efficient design. This can be illustrated by a chain of similar links. Making any link stronger, and thereby heavier than the weakest link would be wasteful. The philosophy of different

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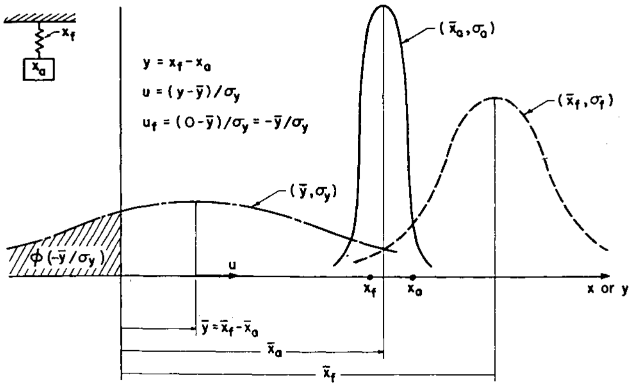



**Fig. 1 Probability of failure.**

safety factors has been recognized to some extent by the introduction of additional safety factors for such items as castings and fittings.

It is hoped that this paper will promote a better understanding and a more logical approach to the structural design problem. Procedures are presented to establish or estimate the proper safety factors from data, which is available or can be obtained by the designer.

The components of the structure can be visualized as springs in parallel and series with respect to the applied loads. Each group of parallel elements can be considered as an equivalent series element. Failure of a parallel element results in a redistribution of the load among the remaining parallel elements of the group, whereas failure of a series element results in failure of the structure. The design of efficient structures with elements in series is discussed in Ref. 1. The design of efficient structures with special types of parallel elements is discussed in Ref. 2.

One of the questions to be resolved is how to translate a probability of failure into an equivalent factor of safety. This requires a knowledge of the density or frequency distribution of the applied and failing strengths. The properties of independent and dependent variates are discussed below.

The following definitive formulas, available in various texts, are employed in describing the characteristics of the distributions of a population:

Mean Value

$$\bar{x} = 1/n \; \Sigma \; x_i \tag{1}$$

Variance

$$\sigma^2 = 1/n \; \Sigma \; (\delta x_i)^2 \tag{2a}$$

where

$$\delta x_i = x_i - \bar{x} \tag{2b}$$

The coefficient of variance

$$\gamma = \sigma/\bar{x} \tag{3}$$

is usually easier to estimate than the variance. Past experience on similar types of structures and material can be employed for the estimate.

The frequency of the normal (gaussian) distribution

$$f(x) = [1/(2\pi\sigma^2)^{1/2}] \exp[-(x - \bar{x})^2/2\sigma^2] \tag{4}$$

is defined in terms of the mean value ($\bar{x}$) and the variance ($\sigma^2$).

The probability of finding a value $x$ between two given values $x_0$ and $x_1$ is

$$Pr(x_0 \leq x \leq x_1) = \int_{x_0}^{x_1} f(x)dx = \int_{-\infty}^{x_1} f(x)dx - \int_{-\infty}^{x_0} f(x)dx \tag{5a}$$

$$Pr(x_0 \leq x \leq x_1) = \phi\left[\frac{(x_1 - \bar{x})}{\sigma}\right] - \phi\left[\frac{(x_0 - \bar{x})}{\sigma}\right] = \phi(u_1) - \phi(u_0) \tag{5b}$$

where

$$u_i = (x_i - \bar{x})/\sigma \tag{5c}$$

Transformation laws are utilized often to estimate the distribution of a dependent function from the distribution of the independent variates. On other occasions a transformation of a variate may result in a distribution that is better suited to the design assumption of normal distributions (e.g., fatigue failure as a function of $\log N$ rather than $N$).

In general, a function can be expanded about the mean value of the variates by the use of the Taylor Series Expansion, if one assumes continuity of the function and its derivatives. That is,

$$F = F(x_1, x_2, \ldots x_i, \ldots x_m) = F(\bar{x}_i) + \sum_{j=1}^{\infty} \frac{1}{j!}\left[\sum_{i=1}^{m} \delta x_i \left(\frac{\partial}{\partial x_i}\right)\right]^j F \tag{6a}$$

and

$$\delta F = F - F(\bar{x}_i) = \sum^{\infty} \frac{1}{j!}\left[\sum^{m} \delta x_i \left(\frac{\partial}{\partial x_i}\right)\right]^j F \tag{6b}$$

By definition

$$\bar{F} = \frac{1}{n}\sum_{k=1}^{n} F_k = \frac{1}{n}\left[nF(\bar{x}_1) + \sum^{m}\left(\frac{\partial F}{\partial x}\sum^{n}\delta x_{ik}\right) + \frac{1}{2}\sum^{m}\left(\frac{\partial^2 F}{\partial x_i \partial x_l}\sum^{n}\delta x_{ik}\delta x_{lk}\right) + \text{higher order derivatives}\right] \tag{6c}$$

For independent variates ($x_i$ and $x_l$), we have by definition

$$\frac{1}{n}\sum^{n}\delta x_{ik} = 0 \tag{6d}$$

$$\frac{1}{n}\sum^{n}(\delta x_{ik})^2 = \sigma_{x_i}{}^2 \tag{6e}$$

and

$$\frac{1}{n}\sum^{n}(\delta x_{ik})(\delta x_{lk}) = 0 \tag{6f}$$

Substituting Eqs. (6d–6f) into Eq. (6c) and assuming that higher order derivates can be neglected with respect to lower order derivates, results in

$$\bar{F} \sim F(\bar{x}_i) + \frac{1}{2}\sum^{m}\sigma_{x_i}{}^2 \frac{\partial^2 F}{\partial x_i{}^2} \; (@ \; x_i = \bar{x}_i) \tag{6g}$$

Similarly since

$$\sigma_F = \frac{1}{n}\sum^{n}(\delta F)^2 \tag{6h}$$

we obtain from Eqs. (6b, 6d–6f)

$$\sigma_F{}^2 \sim \sum^{m}\sigma_{x_i}{}^2\left(\frac{\partial F}{\partial x_i}\right)^2 \; (@ \; x_i = \bar{x}_i) \tag{6i}$$

The distribution of the function can be different from that of the independent variates, but for the purpose of employing sampling data for engineering design we will assume all of the distributions are sufficiently normal.

To obtain information of the population of a dependent variate, it is not absolutely necessary to know the exact functional relationship between the function and independent variates. Erroneous but satisfactory empirical relationships can be assumed, and the available experimental data can be analyzed to obtain least square values and confidence levels relative to the empirical relationship assumed (see examples and Appendix).

The transformation equations are utilized in determining the probability of failure. Let ($\bar{x}_a$, $\sigma_a$) and ($\bar{x}_f$, $\sigma_f$) in Fig. 1 represent the normal distributions of the applied ($x_a$) and failing ($x_f$) strength indices, respectively. These indices are not necessarily associated with the load but may be any quantity, which defines the applied and failing populations. Some other examples are stress or cumulative damage. Methods of determining these distributions from the available data and transformation laws will be illustrated in the examples.

The probability of failure is equivalent to the probability that the applied index will exceed the failing index. Let us examine the density distribution for $y = x_f - x_a$, utilizing the transformation laws (Eqs. 6), which will express the characteristic properties of the $y$ distribution in terms of the characteristic properties of the $x_f$ and $x_a$ distributions.

If

$$y = x_f - x_a \tag{7a}$$

then

$$\bar{y} = \bar{x}_f - \bar{x}_a \qquad \text{[see. Eq. (6g)]} \tag{7b}$$

and

$$\sigma_y = (\sigma_f^2 + \sigma_a^2)^{1/2} \qquad \text{[see. Eq. (6i)]} \tag{7c}$$

It is therefore possible to plot the frequency distribution of $y$ as shown in Fig. 1. The probability that $y = x_f - x_a \leq 0$ is the probability of failure and is equal to the area of the $y$ curve to the left of the ordinate axis. It is evident from Fig. 1 that the ordinate axis is located $-\bar{y} = -(\bar{x}_f - \bar{x}_a)$ from the mean value of the $y$ distribution. Therefore,

$$Pr(x_f - x_a \leq 0) = \phi(-\bar{y}/\sigma_y) = \phi(u_f) \qquad \text{[see. Eq. (5)]} \tag{8a}$$

Employing the relationships of Eqs. (7b) and (7c), we obtain

$$u_f = -(\bar{y}/\sigma_y) = [-(\bar{x}_f - \bar{x}_a)/(\sigma_f^2 + \sigma_a^2)^{1/2}] \tag{8b}$$

Operating on Eq. (8b) by dividing both numerator and denominator of the right-hand side by $\bar{x}_a$ and regrouping results in

$$\text{Factor of safety} = \frac{\bar{x}_f}{\bar{x}_a} = r = \frac{1 + |\xi|(1 + a^2 - a^2\xi)^{1/2}}{1 - \xi^2} \tag{9a}$$

where

$$\xi = u_f\gamma_f \tag{9b}$$

$$\gamma_a = \sigma_a/\bar{x}_a \tag{9c}$$

$$\gamma_f = \sigma_f/\bar{x}_f \tag{9d}$$

and

$$a = \gamma_a/\gamma_f \tag{9e}$$

Equations (7–9) are applicable to any function, which is the difference of two independent variates. The function and the variates can have different types of distributions. The normal distribution is assumed in the design procedure, since it satisfactorily approximates most structural distributions and can be adapted readily to limited sampling data because of the many pertinent mathematical relationships (e.g., $\phi$, $\chi^2$, $t$, and $F$ functions) that have been derived and tabulated. It is never possible to obtain complete information about a distribution from a limited sample. It is therefore expedient to assume a normal distribution in analyzing the sample and to obtain the pertinent parameters defining the normal distribution with some known degree of confidence. Designs based upon normal distributions with large deviations from the mean, however, should be viewed only as a mathematical exercise, indicative but not representing an actual practical design.

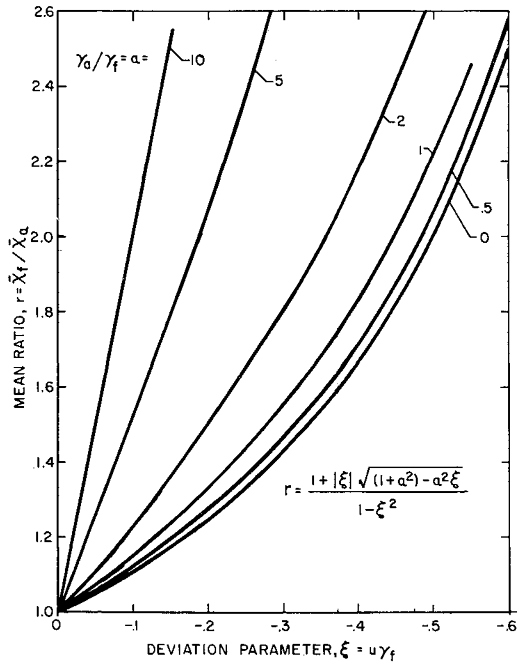



**Fig. 2 Required mean ratio.**

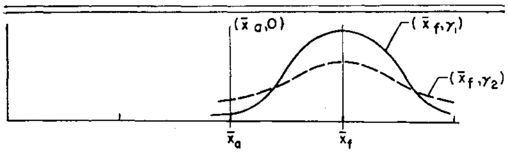
Figure 2 is a graphical representation of Eq. (9a) and indicates that the required safety factor increases with higher structural reliability and variabilities. Lowest values of $r$ occur when $a = 0$ and correspond to an invariant applied index as may occur in a proof test of some sample specimens. Thus the probability of failure in service is at least as great as obtained from a limit load proof test. The desired structural reliability can only be demonstrated by loading above the limit load. Typical applications of Eq. (9a) are presented in Tables 1–4 which were taken from examples presented in Ref. 3. Table 1 indicates that the probability of failure increases with the variability of the failure index ($\gamma_f$). Table 2 indicates that the probability of failure increases with the variability of the applied index ($\gamma_a$). Table 3 indicates that the probability of failure increases rapidly with decreasing factors of safety. This characteristic permits the selection of a few critical loading conditions and components in evaluating a structural design, since all other considerations would result in insignificant probabilities of failure. Analyzing every component for every loading condition would be unnecessary provided the corresponding variabilities are not excessive. Table 4 indicates that the present design procedure of applying an arbitrary factor of 1.5 upon conservative estimates (90% confidence) of the applied and failing indices may not be satisfactory for designs of high variability and structural reliability and may be too heavy for designs of low variability and structural reliability. The design problem thus resolves itself into defining the distributions of the applied and failing indices as well as an acceptable probability of failure.

The applied index is usually obtained by analyses utilizing applicable relationships and transformation formulations of the expected variations. The results are supported, wherever possible, by flight and structural tests on similar designs or models.

The failing index also employs applicable formulations but depends very strongly upon failure data from pertinent experimental programs, since the failure phenomenon is rarely adequately understood. In many cases, such as a new material, the data are too meager to determine distribution, and extrapolation of similar data is often necessary.

The probability of failure is the complement of the structural reliability, which is defined herein as the probability

**Table 1 Effect of failure variability**




$r = 1.5 = \frac{\bar{x}_f}{\bar{x}_a}$ ; $\gamma_a = 0 \therefore \xi = -.335$

| FAILURE VARIABILITY $\gamma_f$ | PROBABILITY OF FAILURE $\phi(u)$ |
|---|---|
| 0 | 0 |
| .06 | < $10^{-7}$ |
| .09 | .0000996 |
| .12 | .002635 |
| .15 | .01287 |

that the structure will perform its assigned tasks. The structural reliability selected should be a satisfactory compromise between an economic and feasible design as opposed to extreme safety with resultant overweight and degradation of performance. Some of the factors to be considered in arriving at the optimum structural reliability are discussed in Ref. 4.

The technique will be illustrated in the following examples. It should be noted that all of the data were assumed in the illustrative problems and should not be construed as representative of actual data. In addition, dimensional tolerances are assumed to represent three standard deviations.

## Example 1

Design a tension cover of a titanium box beam given the following information:

1) Load per inch $\bar{N} = 10{,}000 \pm 1000$ lb/in. (from load and stress analysis); therefore, $\bar{N} = 10{,}000$, $\sigma_N = 333$.

2) The area can vary 10%, i.e., $A = \bar{A} \pm 0.1\bar{A}$; therefore, $\gamma_A = 0.033 = 0.1\bar{A}/3\bar{A}$.

3) Ultimate strength at 50°F is $S_{RA} = 140{,}000$ psi ($A$ value = 99% probability); $S_{RB} = 150{,}000$ psi ($B$ value = 90% probability). Limited tests at temperature indicated that strength at temperature $S_T = S_R - C_1(T - 50) - C_2(T - 50)^2$ with $\bar{C}_1 = 100$ and $\sigma_{C_1} = 5$ psi/°F and $\bar{C}_2 = 0.1$ and $\sigma_{C_2} = 0.003$ psi/(°F)$^2$ obtained from least square analysis of available data (Appendix A).

4) Temperature $T = 450 \pm 45$°F; therefore $\overline{T - 50} = 400$; $\sigma_T = 15$.

5) Probability of failure $\phi(u) = 0.01$; therefore, $u = -2.327$.

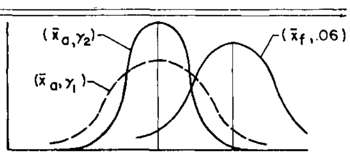
### Applied Stress Index

$$x_a = S_a = N/A \tag{10a}$$

**Table 2 Effect of applied variability**

$r = 1.5$

$\gamma_f = .06$




| APPLIED VARIABILITY $\gamma_a$ | PROBABILITY OF FAILURE $\phi(u)$ |
|---|---|
| 0 | < $10^{-7}$ |
| .06 | .000002 |
| .12 | .000419 |
| .30 | .0548 |
| .60 | .2047 |

$$\bar{x}_a = \bar{N}/\bar{A} = 10{,}000/\bar{A} \tag{10b}$$

$$\gamma_a = (\gamma_N^2 + \gamma_A^2)^{1/2} = \left[\left(\frac{1000}{3(10{,}000)}\right)^2 + (0.033)^2\right]^{1/2} = 0.047 \tag{10c}$$

### Failing Stress Index

From the properties of the normal distribution we can obtain

$$\bar{S}_R = S_{RB} + 1.237(S_{RB} - S_{RA}) \text{ (Ref. 1)} \tag{11a}$$
$$= 162{,}370 \text{ psi}$$

$$\sigma_{SR} = (S_{RB} - S_{RA})/\,1.04 \text{ (Ref. 1)} \tag{11b}$$
$$= 9600 \text{ psi}$$

Since

$$S_T = S_R - C_1(T - 50) - C_2(T - 50)^2 \tag{12a}$$

From the transformation laws of Eq. (6) we obtain

$$\bar{S}_T = \bar{S}_R - \bar{C}_1(\overline{T - 50}) - \bar{C}_2(\overline{T - 50})^2 + \bar{C}_2\sigma_T^2 \tag{12b}$$
$$\bar{x}_f = S_T\dagger = 106{,}350 \text{ psi}$$

and

$$\sigma_{ST} = \{\sigma_{SR}^2 + \sigma_{C_1}^2(\overline{T - 50})^2 + \sigma_{C_2}^2(\overline{T - 50})^4 + \sigma_T^2[\bar{C}_1 + 2\bar{C}_2(\overline{T - 50})^2]\}^{1/2} \tag{12c}$$
$$\sigma_f = \sigma_{ST}\dagger = 10{,}000 \text{ psi}$$

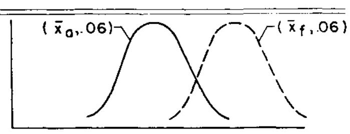
therefore,

$$\gamma_f = \sigma_f/\bar{x}_f = 0.094 \text{ [see Eq. (9d)]}$$
$$a = \gamma_a/\gamma_f = 0.5 \text{ [see Eq. (9c)]}$$
$$\xi = u\gamma_f = -0.218 \text{ [see Eq. (9b)]} \tag{12d}$$
$$r = 1.31 = \bar{x}_f/\bar{x}_a = 106{,}350/(10{,}000/\bar{A}) \text{ [see Eq. (9a)]}$$
$$A = r\bar{N}/\bar{S}_T = 1.31\ (10{,}000)/106{,}350 = 0.123$$

Various design approaches indicating the increase in weight and structural reliability with arbitrary safety factors and conservative estimates of the design parameters are compared in Table 5.

**Table 3 Effect of safety factor**

$\gamma_a = \gamma_f = .06$




| SAFETY FACTOR $r$ | PROBABILITY OF FAILURE $\phi(u)$ |
|---|---|
| 1.00 | .5000 |
| 1.10 | .1190 |
| 1.20 | .0139 |
| 1.30 | .00107 |
| 1.40 | .0000544 |
| 1.50 | .00000192 |

† The estimate of the failing population for the design condition is obtained by considering a condition for which adequate statistical data exists (room temperature) together with other conditions for which limited data exists (strength vs temperature data).

## Example 2

Design a monocoque titanium cylinder for compression given the following information:

1) Load per inch $\bar{N}$ = 10,000 ± 1000 lb./in.; therefore, $\bar{N}$ = 10,000 and $\sigma_N$ = 333.

2) The thickness $t$ can vary 3%; therefore, $\gamma_t = 0.01$.

3) The radius $R$ = 100 ± 1 in.; therefore, $\bar{R}$ = 100 and $\sigma_R = 0.33$.

4) Modulus: $E_T = E_0 - C_3 T$, $\bar{E}_0 = 16.0$ (10) psi, $\sigma_{E_0} = 0.1\ (10)^6$ psi, $\bar{C}_3$ = 7000 psi/°F, and $\sigma_{C_3}$ = 1000 psi/°F.

5) Stability: A log-log plot of $S_f/E$ vs $t/R$ from experimental data suggest the approximate solution $\log(S/E) = m \log(t/R) + \log C$. A least square analysis of the data (Appendix A) results in $\overline{\log C} = 0.980$, $\sigma_{\log C} = 0.040$, $\bar{m}$ = 1.60, and $\sigma_m = 0.10$.

6) Temperature $T$ = 800 ± 50°F; therefore, $\bar{T}$ = 800 and $\sigma_T = 16.7$.

7) Probability of failure $\phi(u) = 0.01$; therefore, $u = -2.327$.

**Applied Stress Index**

$$x_a = S = N/t \qquad \text{[see Eq. (10a)]}$$

$$\bar{x}_a = \bar{N}/\bar{t} = 10{,}000/\bar{t} \qquad \text{[see Eq. (10b)]}$$

$$\gamma_a = (\gamma_N^2 + \gamma_A^2)^{1/2} = [(0.033)^2 + (0.01)^2]^{1/2} = 0.0346 \quad \text{[see Eq. (10c)]}$$

**Failing Stress Index** [see Eq. (6, 9)]

***Stability coefficient***

$$\left.\begin{aligned} \bar{C} &= \{1 + [(2.3)^2/2]\sigma_{\log C}^2\} \log^{-1} [\overline{\log C}] \\ \bar{C} &= (1.0042)\ 9.55 = 9.59 \end{aligned}\right\} \qquad (13a)$$

$$\left.\begin{aligned} \sigma_C &= 2.3\sigma_{\log C} \log^{-1}[\overline{\log C}] \\ \sigma_C &= (2.3)(0.04)(9.55) = 0.88 \\ \gamma_C &= 0.0918 \end{aligned}\right\} \qquad (13b)$$

***Thickness ratio***

$$F = (t/R)^m \qquad (14a)$$

$$\bar{F} = (\bar{t}/\bar{R})^{\bar{m}} + \tfrac{1}{2}\{m(m+1)\sigma_t^2(\bar{t}^{m-2}/\bar{R}^m) + m(m+1)\sigma_R^2(\bar{t}^m/\bar{R}^{m+2}) + \sigma_m^2(\bar{t}/\bar{R})^m[(1/m) + \ln^2 m]\} \qquad (14b)$$

$$\bar{F} = (\bar{t}/100)^{1.6}[1 + \tfrac{1}{2}(0.0096 + 0.00042 + 0.0084)] = 1.0092(\bar{t}/100)^{1.6}$$

$$\sigma_F = \{\sigma_t^2\ (m\bar{t}^{m-1}/\bar{R}^m)^2 + \sigma_R^2\ (-m\bar{t}^m/R^{m+1})^2 + \sigma_m^2\ [\ln m\ (t^m/R^m)]^2\}^{1/2} \qquad (14c)$$

$$\sigma_F = (\bar{t}/100)^{1.6}\ [0.000256 + 0.000256 + 0.0022]^{1/2} = 0.052(\bar{t}/100)^{1.6}$$

$$\gamma_F = \frac{\sigma_F}{\bar{F}} = \frac{0.052}{1.0092}\,\frac{(\bar{t}/100)^{1.6}}{(\bar{t}/100)^{1.6}} = 0.0516$$

***Modulus***

$$E_T = E_0 - C_3 T \qquad (15a)$$

$$\bar{E}_T = \bar{E}_0 - \bar{C}_3\bar{T} \qquad (15b)$$

$$\bar{E}_T = 10.4(10)^6 \text{ psi}$$

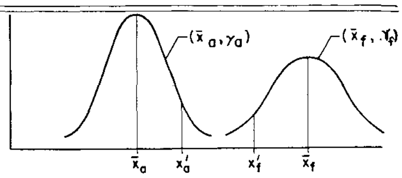
$$\sigma_{E_T} = [\sigma_{E_0}^2 + \sigma_{C_3}^2\bar{T}^2 + \sigma_T^2\bar{C}_3^2]^{1/2} \qquad (15c)$$

$$= 0.815\ (10)^6$$

$$\gamma_{ET} = 0.0785$$

**Table 4 Effect of probability (90% confidence).**

$$r' = \frac{x_f'}{x_a'} = \frac{\bar{x}_f - 1.286\,\sigma_f}{\bar{x}_a + 1.286\,\sigma_a}$$




| $\gamma_f = .1$ | $\phi(u) = 10^{-4}$, $u = -3.719$ | | $\phi(u) = 10^{-6}$; $u = -4.75$ | |
|---|---|---|---|---|
| $\gamma_a$ | $r = \bar{x}_f/\bar{x}_a$ | $r' = x_f'/x_a'$ | $r = \bar{x}_f/\bar{x}_a$ | $r' = x_f'/x_a'$ |
| 0 | 1.59 | 1.385 | 1.90 | 1.64 |
| .05 | 1.64 | 1.342 | 1.96 | 1.61 |
| .10 | 1.74 | 1.350 | 2.11 | 1.64 |
| .20 | 2.06 | 1.430 | 2.52 | 1.75 |

***Stress***

$$\bar{S}_f = \bar{C}\bar{E}\ \overline{(t/R)^m} \qquad (16a)$$

therefore,

$$\bar{x}_f = \bar{S}_f = 6.24\ (10)^4\bar{t}^{1.6}$$

$$\gamma_f = [\gamma_C^2 + \gamma_E^2 + \gamma_F]^{1/2} = 0.132 \qquad (16b)$$

Thus the estimate of the failing population is obtained by evaluating all of the experimental and analytical data, which is pertinent in defining the failure population. The design procedure is then conducted in the customary way after the applied and failing stress populations are satisfactorily defined.

$$a = \gamma_a/\gamma_f = 0.0346/0.132 = 0.262$$

$$\xi = u\gamma_f = (-2.327)(0.132) = -0.307$$

$$r = 1.46 \text{ (see Fig. 2)}$$

$$r = \bar{x}_f/\bar{x}_a = 6.24(10)^4\bar{t}^{1.6}/[(10)^4/\bar{t}] = 1.46$$

$$\bar{t} = (1.46/6.24)^{1/2.6} = 0.573$$

$$\bar{S}_f = r\bar{N}/\bar{t} = 1.46\ (10{,}000)/0.573 = 25{,}500 \text{ psi (elastic)}$$

Table 6 indicates the design and unreliability of cylinders designed with the classical formula $S_f = C_4E\ (t/R)$ and the arbitrary safety factor of 1.5. Mean values of the variables were employed in the calculations, although several values of the stability constant were assumed. The applied and failing variabilities were assumed to be identical to those calculated in the preceding example. The table indicates that the probability of failure increases with higher stability constants resulting in higher instability stresses with resulting lower design thicknesses.

## Example 3: Confidence in Design

The design is based upon applied and failing index distributions obtained from a limited sampling of the populations.

**Table 5 Probability of failure for various design approaches**

| Design approach | Safety factor, $r$ or $r'$ | Area | Probe of failure |
|---|---|---|---|
| Statistical $A = \dfrac{r\bar{N}}{\bar{x}_f}$ | 1.31 | 0.123 | 0.01 |
| Arbitrary safety factor on means, $\bar{A} = r'\bar{N}/\bar{x}_f$ | 1.50 | 0.141 | 0.00021 |
| | 1.865 | 0.175 | 0.000001 |
| Arbitrary safety factor on extremes, 0.9 $= 1.1\,\dfrac{\bar{N}r'}{x_f'}$ | 1.50 ($x_{f(3\sigma)}'$ = 85,700 psi) | 0.214 | $<10^{-8}$ |

**Table 6 Probability of failure for various designs**

| Stability coeff. | Thickness | Mean ratio | Prob. of failure |
|---|---|---|---|
| $C_4 = S_f/E(t/R)$ | $^a t = (1.5\ \bar{N}\bar{R}/C_4\bar{E})^{1/2}$ | $r = \bar{S}_f/\bar{S}_a$ | $\phi(u)$ |
| 0.6 | 0.49 | 1.25 | 0.0749 |
| 0.4 | 0.60 | 1.53 | 0.00554 |
| 0.2 | 0.845 | 2.16 | 0.000029 |
| 0.44 | 0.573[b] | 1.46 | 0.01 |

[a] Design based on arbitrary saftey factor of 1.5 with stability equation $S_f = C_4E(t/R)$.
[b] Design identical to that which results from the statistical analysis $S_f = 9.59\ E(t/R)^{1.6}$ (see Example 2).

The only way positively to know a distribution is to test the entire population. This is hardly ever practical and would defeat any need for statistical data. Design values are based upon a sampling of the population. There is a possibility that a sampling would result in an optimistic representation of the actual population. It is therefore necessary to reduce the experimental determined estimates of the mean and standard deviation values to obtain a high degree of confidence in the design.

The adequacy of the design is evidenced by the resulting strength of the specimens built to the design specifications. A failing strength equal to the strength predicted by analysis would indicate a confidence of 0.50 that the structure was designed with the correct failure distribution and resulting probability of failure. A higher or lower strength would indicate correspondingly higher or lower confidence in the failure population utilized in the design. If confidence in the accuracy of the failure population is not deemed sufficiently adequate, then either the structure must be redesigned with a more conservative estimate of the failure population that has the required confidence, or the present design must be acceptable with the higher probability of failure that would result from this more conservative estimate.

Tests are employed to increase knowledge of the population or a subset characterized by the design. Increased confidence in design values can be obtained by increasing the number of tests ($n$), although the effect becomes smaller for large $n$, or by conservatively estimating the design values.[3] Design values in Ref. 5 are usually presented with a high degree of confidence. As an example, paragraph 3.1.1.1.1 indicates that the $A$ and $B$ values represent a 0.95 confidence that these values will be exceeded 99% and 90% of the time, respectively. This implies that $x_A = \bar{x} - 2.326\sigma$, and $x_B = \bar{x} - 1.286\sigma$, where $\bar{x} \geq \tilde{x} - \{st[n-1, \phi(u_c)]/(n-1)^{1/2}\}$; $\sigma \leq s(n/\chi^2_{n-1,\ \phi(u_c)})^{1/2}$; $\tilde{x}$ = mean value obtained from $n$ tests; $\bar{x}$ = mean value of population; $s$ = standard deviation obtained from $n$ tests; $\sigma$ = standard deviation of population; $t[n-1, \phi(u_c)]$ = Student's $t$ function [tabulated for $n - 1$ and $\phi(u_c)$]; $\chi^2_{n-1,\ \phi(u_c)}$ = chisquared function [tabulated for $n - 1$ and $\phi(u_c)$]; and $C = 2\phi(u_c) - 1$ = Confidence equal to the probability of placing proper limits on a variable [for $C = 0.95$, $\phi(u_c) = 0.975$, and $u_c = 1.96$].

Let us consider the design and testing of a structure designed for fatigue utilizing an arbitrary cumulative damage theory. The following was ascertained:

1) Analysis of experimental data on similar structures with the cumulative damage theory resulted in an estimate of the failure index with a mean value $\bar{D}_f = 0.800$ and a coefficient of variability $\gamma_f = 0.10$. This was employed in designing the structure.

2) The loading history and variability of the material and elemental fatigue data indicated a coefficient of variability of $\gamma_a = 0.05$.

3) The structure was designed for a limit stress level, which resulted in $\bar{D}_a = 0.490$ and $r = 1.63$. This corresponded to the desired probability of failure of $\phi(u_f) = 0.0001$.

4) Three specimens were fabricated and tested in fatigue.

5) A representative loading history approximating a percentage (5%) of the total design life of the structure was repeated until the specimens failed. The failures corresponded to $R_i\bar{D}_a$ where $R_i = 1.55$, 1.65, and 1.75. This resulted in an average test factor of $\bar{R} = (1/n)\Sigma R_i = 1.65$.

We wish to determine the following, assuming that the coefficient of variability is satisfactorily defined from previous experience:

1) With what confidence can we say that the structure has a probability of failure of $a = 0.0001$, $b = 0.001$, and $c = 0.01$?

2) What should the value of test factor $\bar{R} = (1/n)\Sigma R_i$ have been in order to predict a probability of failure of 0.0001 with a confidence of $a = 0.75$, $b = 0.90$, and $c = 0.99$?

3) How would the confidence change if the number of specimens increased from 3 to $a = 5$ specimens, $b = 9$ specimens, and $c = 12$ specimens, whereas $\bar{R}$ remained at 1.65?

The results of the analysis of the test data is presented in Table 7. The analysis indicates that very little can be done to increase the confidence in the design, as evidenced by the test results, unless one is prepared to strengthen the structure, reduce the loadings, or accept a higher probability of failure.

In conclusion, the following points can be made:

1) Factors of safety can be put on a more logical basis when one considers the variability of the applied chronological data

**Table 7 Analysis of test data**

| Prob. of failure, $\phi(u_f)$ | Failure deviations, $u_f$ | Mean design ratio, $r$ | Mean test ratio, $\bar{R}$ | No. of spec., $n$ | Confidence deviations, $u_c$ | Minimum confidence, $C$ |
|---|---|---|---|---|---|---|
| | | Confidence for probability of failure, $[(\bar{R}/r) - 1]n^{1/2}/\gamma_f$ | | | | |
| 0.0001 | −3.715 | 1.63 | 1.65 | 3 | 0.213 | 0.584 |
| 0.001 | −3.090 | 1.48 | 1.65 | 3 | 1.989 | 0.9766 |
| 0.01 | −2.326 | 1.33 | 1.65 | 3 | 4.167 | 0.99998 |
| | | Scatter factor for given confidence, $r[(u_c\gamma_f/n^{1/2}) + 1]$ | | | | |
| 0.0001 | −3.715 | 1.63 | 1.650 | 3 | 0.213 | 0.584 |
| 0.0001 | −3.715 | 1.63 | 1.694 | 3 | 0.675 | 0.75 |
| 0.0001 | −3.715 | 1.63 | 1.751 | 3 | 1.282 | 0.90 |
| 0.0001 | −3.715 | 1.63 | 1.849 | 3 | 2.326 | 0.99 |
| 0.0001 | −3.715 | 1.63 | 1.63 | 3 | 0 | 0.50 |
| | | Effect of number of specimens on confidence, $[(\bar{R}/r) - 1]n^{1/2}/\gamma_f$ | | | | |
| 0.0001 | −3.715 | 1.63 | 1.65 | 3 | 0.213 | 0.584 |
| 0.0001 | −3.715 | 1.63 | 1.65 | 6 | 0.3048 | 0.618 |
| 0.0001 | −3.715 | 1.63 | 1.65 | 9 | 0.3681 | 0.644 |
| 0.0001 | −3.715 | 1.63 | 1.65 | 12 | 0.4250 | 0.665 |

**Table 8 Data**

| Test | Temp., °F, $x$ | Modulus, $10^6$ psi, $E$ | $y = (E - 10^7)/10^6$, [a]$y$ | $xy$ | $x^2$ | $y^2$ |
|---|---|---|---|---|---|---|
| 1 | 30 | 16.42 | 6.42 | 182.6 | 900 | 41.2164 |
| 2 | 100 | 16.12 | 6.12 | ... | ... | ... |
| 3 | 200 | 15.65 | 5.65 | ... | ... | ... |
| 4 | 300 | 15.13 | 5.13 | ... | ... | ... |
| 5 | 400 | 14.76 | 4.76 | ... | ... | ... |
| 6 | 500 | 14.33 | 4.33 | ... | ... | ... |
| 7 | 600 | 13.89 | 3.89 | ... | ... | ... |
| 8 | 700 | 13.47 | 3.47 | ... | ... | ... |
| 9 | 800 | 13.03 | 3.03 | ... | ... | ... |
| 10 | 900 | 12.51 | 2.51 | ... | ... | ... |
| 11 | 1000 | 12.01 | 2.01 | ... | ... | ... |
| 12 | 1100 | 11.40 | 1.40 | ... | ... | ... |
| 13 | 1200 | 11.00 | 1.00 | ... | ... | ... |
| 14 | 1300 | 10.73 | 0.73 | ... | ... | ... |
| 15 | 1400 | 10.24 | 0.24 | ... | ... | ... |
| 16 | 1500 | 9.99 | −0.01 | ... | ... | ... |
| Σ | 12,030 | 210.68 | 50.68 | 23,008.60 | 12,400,900 | 228.5614 |

[a] Note that the value of $E$ was transformed to $y$ to simplify the calculations by reducing the variables to the first significant figure that changes.

and the probabilistic distribution of the failing population. Arbitrary factors of safety can result in unnecessary weight or inadequate reliability.

2) A more consistent design procedure would result if mean rather than extreme values were employed in designing a structure. The factors of safety employed would incorporate the variability of the pertinent design and test parameters.

3) Statistical techniques can be employed to convert a structural reliability requirement to an equivalent factor of safety problem. Conversely a given design or test data can be analyzed for confidence.

4) Statistical techniques offer logical procedures for extrapolating to the limits of available test data. Empirical relationships can be assumed and the available data analyzed to obtain satisfactory approximations to the solution.

5) The designer should be cognizant of all of the design philosophies and capable of translating them into a logical design procedure to secure a desired structural reliability.

## Appendix: Statistical Determination of Functional Relationship from Tests

Let us assume that a functional relationship exists between two variables, which by a suitable transformation can be expressed in a linear form. Many relationships, linear and nonlinear, can be expressed as $y = mx + b$. Thus, in example 2, the stability of a cylinder is expressed as $S = CE(t/R)^m$. Letting $\log S/E = y$, $\log(t/R) = x$, and $\log C = b$ results in $y = mx + b$.

A plot of experimental data, which determines values of $y$ from known values of $x$ would indicate if the relationship is approximately linear. A crude approximation of the linear relationship could be made by eye, but the result would depend upon and vary with the individual. We must have some measure on how well and with what certainty the line does fit the data. A least square technique is employed to

**Table 9 Calculations**

| Item | Symbol | Source | Example |
|---|---|---|---|
| 1 | $n$ | Data | 16 |
| 2 | $\Sigma x_i$ | Data | 12,030. |
| 3 | $\bar{x}$ | 2/1 | 751.875 |
| 4 | $\bar{x}\Sigma x_i$ | 2 × 3 | 9,045,056.25 |
| 5 | $\Sigma(x_i)^2$ | Data | 12,400,900. |
| 6 | $\Sigma y_i$ | Data | 50.68 |
| 7 | $\bar{y}$ | 6/1 | 3.1678 |
| 8 | $\bar{y}\Sigma y_i$ | 6 × 1 | 160.5289 |
| 9 | $\Sigma(y_i)^2$ | Data | 228.5614 |
| 10 | $n\bar{x}\bar{y}$ | 1 × 3 × 7 | 38,105.025 |
| 11 | $\Sigma x_i y_i$ | Data | 23,008.60 |
| 12 | $s_{xx}$ | 5–4 | 3,355,843.75 |
| 13 | $s_{yy}$ | 9–8 | 68.0325 |
| 14 | $s_{xy}$ | 11–10 | −15,096.425 |
| 15 | $\bar{m}$ | 14/12 | −0.00449854 |
| 16 | $\bar{b}$ | 7–15 × 3 | 6.549846 |
| 17 | $(s_{xy})^2/s_{xx}$ | 14 × 15 | 67.91199614 |
| 18 | $\sigma_y^2$ | [8–17]/[1–2] | 0.00860741857 |
| 19 | $\sigma_y$ | $18^{1/2}$ | 0.09227617 |
| 20 | $\sigma_m^2$ | 18/12 | $0.25649045\ (10)^{-8}$ |
| 21 | $\sigma_m$ | $20^{1/2}$ | $0.5065\ (10)^{-4}$ |
| 22 | $\sigma_b^2$ | $18[(1/1) + (3^2/12)]$ | 0.0019879452 |
| 23 | $\sigma_b$ | $22^{1/2}$ | 0.04457 |

$\bar{x} = (1/n)\Sigma x_i$

$s_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma(x_i^2) - n\bar{x}^2 = \Sigma(x_i^2) - \bar{x}\Sigma x_i$

$s_{xy} = \Sigma(x - \bar{x})(y - \bar{y}) = \Sigma x_i y_i - n\bar{x}\bar{y}$

$\bar{m} = s_{xy}/s_{xx}$

$\bar{b} = \bar{y} - \bar{m}\bar{x}$

$\sigma_y^2 = (s_{yy}s_{xx} - s_{xy}^2)/(n - 2)(s_{xx})$

$\sigma_m^2 = \sigma_y^2/s_{xx}$

$\sigma_b^2 = \sigma_y^2[(1/n) + (\bar{x}^2/s_{xx})]$

**Table 10 Solution**

| | |
|---|---|
| $y = \bar{b} + \bar{m}x = (E - 10^7)/10^6$ | $E = \bar{B} + \bar{M}\bar{x}$ |
| $y = 6.549846 - 0.00449855x$ | $E = 16.549846(10)^6 - 4498.55\,x$ (50% confidence) |
| $\sigma_y = 0.092276$ | $\sigma_E = 10^6\,\sigma_y = 92{,}276$ |
| $\sigma_m = 0.00005065$ | $\sigma_m = 10^6\,\sigma_m = 50.65$ |
| $\sigma_b = 0.04457$ | $\sigma_B = 10^6\,\sigma_b = 445{,}700$ |

$$E' \geq \bar{B} + \bar{M}x' - u_c\,\sigma_E'$$

$$E' \geq \bar{B} + \bar{M}x' - u_c\,\sigma_E\{1 + (1/n) + [(x' - \bar{x})^2/s_{xx}]\}^{1/2}$$

$$E' \geq 16{,}549{,}846 - 4498.55\,x' - u_c(92276)\left[1 + \frac{1}{16} + \frac{(x' - 751.875)^2}{3{,}555{,}843.75}\right]^{1/2}$$

where

$u_c = 0$ for Confidence $C = 50.0\%$
$u_c = 1$ for Confidence $C = 84.1\%$
$u_c = 1.286$ for Confidence $C = 90.0\%$

select the most probable line, whereas the offsets of the data points from this line are analyzed statistically to establish confidence limits. The least square technique can also be employed to obtain functional relationships of a higher degree.

All of the experimental data can be employed to determine the population of a functional relationship. There is no need to conduct tests at identical values of the independent variable to obtain sufficient statistical data. It is, therefore, more efficient to conduct the tests with large variations, within the range of interest, rather than limited values of the independent variable.

The calculations necessary to obtain the mean value and standard deviations of the coefficients ($m$ and $b$) of the most probable linear relationship ($y = mx + b$) are indicated in the Tables 8–10, which also illustrate the technique for determining the best linear relationship between the modulus ($E$) and temperature ($x$) of a titanium alloy.

## References

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